Systems of Linear Equations

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   Sir Isaac Newton: If I have seen further,
it is by standing on the shoulders of giants.
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1 Linear Equation and Solutions

What is a linear equation? A linear equation can be written in the form

$$a_1x_1 + a_2x_2 + \cdots + a_nx_n = b \tag{1}$$

In the equation, x_1, x_2, ..., x_n are unknowns and a_1, a_2, ..., a_n, and b are constants. The constant a_k is called the *coefficient* of x_k, b is called the *constant term* of the equation.

A solution of the linear equation (1) is a list of values for the unknowns, say $x_1 = k_1$, $x_2 = k_2$, ..., $x_n = k_n$, or written equivalently as a vector $x = (k_1, k_2, \ldots, k_n)$, which on substitution in Eq. 1 *satisfies* the equation. That is

$$a_1k_1 + a_2k_2 + \cdots + a_nk_n = b$$

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Note that in order to avoid subscripts in Eq. 1, we will usually use x, y for two unknowns, x, y, z for three unknowns, and x, y, z, t for four unknowns.

2 System of Linear Equations

A system of linear equations, in short *linear system*, is a collection of linear equations with the same unknowns. In particular, a system of m linear equations L_1, L_2, ..., L_m in n unknowns x_1, x_2, ..., x_n can be written in the *standard form*

$$\begin{aligned} &a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ &a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ &a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m \end{aligned} \tag{2}$$

where the a_{ij} and b_i are constants. The number a_{ij} is the *coefficient* of the unknown x_j in the equation L_i, and the number b_i is the *constant* of the equation L_i. The constants a_{ij} and b_i are usually given.

The system (2) is called an $m \times n$ (read: m by n) system. It is called a *square system* if $m = n$, that is, if the number equations is equal to the number of unknowns.

The system (2) is said to be *homogeneous* if all the constant terms are zero, that is, if $b_1 = 0$, $b_2 = 0$, ..., $b_m = 0$. Otherwise the system is said to be *nonhomogeneous*.

A *solution* (or a *particular solution*) of the system (2) is a list of values for the unknowns or, equivalently, a vector x, that satisfies each of the equations in the system. The set of all solutions of the system is called the *solution set* or the *general solution* of the system.

3 Matrix Notation for Linear System

The system of linear equations (2) can be written in matrix notation as follows.

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \cdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \cdots \\ b_m \end{bmatrix} \tag{3}$$

The matrix equation (3) can be succinctly written as $Ax = b$, where matrix A is the matrix of coefficients (called *coefficient matrix*), x is the column vector of the n unknowns, and b denotes the column vector of the m constants. Finding solution of the matrix equation $Ax = b$ corresponds to finding the column vector x.

We also define a matrix M associated with the Eq. 3 as follows; it is called the *augmented matrix* of the system.

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{bmatrix}$$

The augmented matrix consists of the coefficient matrix plus column of constants in the last column. The augmented matrix completely determines a system of linear equations. It is helpful when solving for vector x using *Gaussian elimination*.

4 Solutions of Linear System

The system of linear equations (2) is said to be *consistent* if it has one or more solutions,and it is said to be *inconsistent* if it has no solution.

Any linear system (2) has (i) a unique solution, (ii) an infinite number of solutions, or (iii) no solution.

Before we proceed to the method for solving a system of linear equations, let us interpret geometrically what we mean by a solution of these equations. At first, we shall take examples in two-dimensions and three- dimensions, and then generalize for n-dimensions.

Example 1 : Consider the following two linear equations in two dimensional space.

$$\begin{aligned} L_1 : &\quad x + 2y = 3 \\ L_2 : &\quad 4x + 5y = 6 \end{aligned} \tag{4}$$

These equations represent straight lines in the $x - y$ plane and they intersect at the point $(x, y) = (-1, 2)$; the values of the point of intersection satisfy both the linear equations in (4) and therefore the pair of values is the solution of the given linear system; the intersection point lies on both the lines (see Fig. 1).

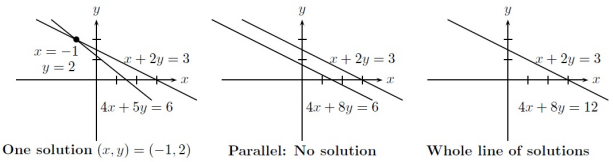

Fig. 1 Example (1) is illustrated graphically. This is called the *row picture* since the horizontal equations are interpreted geometrically. Courtesy Strang (2006).

The solution to the system (4) is obtained by the *elimination algorithm*. Multiply the first equation by -4 and add it to the second equation $(-4L_1 + L_2)$ and use it to replace L_2 to get an *equivalent system*. The equivalent system obtained is in *upper triangular form* (in short, triangular form) and it has the same solution as the original system.

$$\begin{aligned} L_1 : &\quad x + 2y = 3 \\ -4L_1 + L_2 : &\quad -3y = -6 \end{aligned}$$

The solution $y = 2$ and $x = -1$ is obtained by *back-substitution*.

The Eq. 4 is written below in matrix notation. Later in the study, we will obtain solution using matrix formulation.

$$\begin{bmatrix} 1 & 2 \\ 4 & 5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 3 \\ 6 \end{bmatrix} \tag{5}$$

Next, we will consider the following second linear system.

$$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 6 \end{aligned} \tag{6}$$

As seen in Fig. 1, the two lines are parallel and no point lies on both the lines, and hence the linear system has no solution.

Using elimination algorithm to the above system (6), we get the following reduced form.

$$\begin{aligned} L_1 : &\quad x + 2y = 3 \\ -4L_1 + L_2 : &\quad 0 = -6 \end{aligned}$$

The second equation of the equivalent system is *degenerate equation*. The degenerate equation has no solution, so the system has no solution.

Next, we consider the following third linear system.

$$\begin{aligned} x + 2y &= 3 \\ 4x + 8y &= 12 \end{aligned} \tag{7}$$

Here, the two lines coincide and all points, infinite number of them, lie on both the lines; these points are the

solutions of the linear system; thus, the linear system has infinitely many solutions.

Using elimination algorithm to the above system (7), we get the following reduced form.

$$\begin{aligned} L_1: &\quad x + 2y = 3 \\ -4L_1 + L_2: &\quad 0 = 0 \end{aligned}$$

The second equation of the equivalent system is degenerate equation. The degenerate equation has infinite number of solutions. So, all the points of the first equation are the solutions of the linear system.

`Remark:` As we shall see later in the study of linear algebra, the last two cases are *singular* cases: (i) system having no solution and (ii) system having an infinite number of solutions. The system having exactly one solution is the *nonsingular* case; its coefficient matrix has an inverse.

`Solution in three dimensions:` Linear equation with three unknowns, say, x, y, z, represents a plane in three dimensions. Solution to a system of three such equations can be interpreted geometrically as shown in Fig. 2. The three planes may intersect in a point to give unique solution to the system; they may meet in a line or be coincident to give infinitely many solutions; or there may be no point common to all the three planes to give no solution.

`Extending row picture to higher dimensions` (from Strang, 2006): Consider n equations with n unknowns. Each equation still determines a "plane", called a *hyperplane*; it must be flat and extremely thin within n-dimensional space, although it would look solid to us. If a unique solution exits to the linear system, then Strang (2006) explains it as follows. The first equation produces an $(n-1)$-dimensional plane in n dimensions. The second plane intersects it (we hope) in a smaller set of "dimension $n-2$". Assuming all goes well, every new plane (every new equation) reduces the dimension by one. At the end, when all n planes are accounted for, the intersection has dimension zero. It is a *point*, it lies on all the planes, and its coordinates satisfy all n equation. It is the solution!

Example 2 : Solve the following three linear equations with three unknowns using *forward elimination*.

$$\begin{aligned} L_1: &\quad x + y + z = 5 \\ L_2: &\quad x + 2y + 3z = 7 \\ L_3: &\quad x + 3y + 6z = 11 \end{aligned} \tag{8}$$

At first, we eliminate x from L_2 and L_3. We replace L_2 by L_2' with $L_2' = -L_1 + L_2$ and replace L_3 by L_3' with $L_3' = -L_1 + L_3$. These *elimination* operations, also called *elementary* operations yield the following *row-equivalent* system.

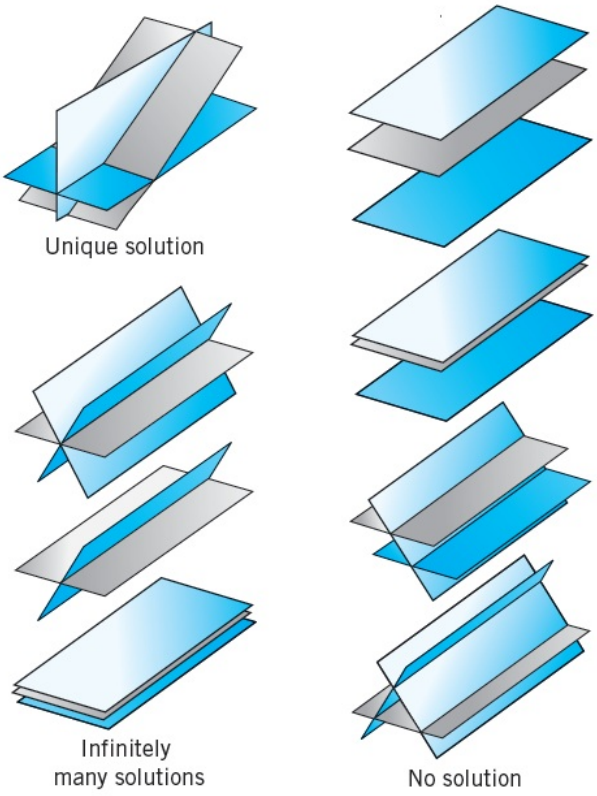

Fig. 2 Solution to three linear equations with three unknowns visualized geometrically. Courtesy Kreyszig *et al.* (2011).

$$\begin{aligned} L_1: &\quad x + y + z = 5 \\ L_2': &\quad y + 2z = 2 \\ L_3': &\quad 2y + 5z = 6 \end{aligned} \tag{9}$$

Then, we eliminate y from L_3'. We replace L_3' by L_3'' with $L_3'' = -2L_2' + L_3'$. The original system gets changed to equivalent *upper triangular* form.

$$\begin{aligned} L_1: &\quad x + y + z = 5 \\ L_2': &\quad y + 2z = 2 \\ L_3'': &\quad z = 2 \end{aligned} \tag{10}$$

On *back-substitution*, we get $z = 2$, $y = -2$ and $x = 5$.

`Remark:` In the triangular form (10), the first unknown x is the leading unknown in the first equation, the second unknown y is the leading unknown in the second equation, and so on. Thus, in particular, the system is *square* and each leading unknown is *directly* to the right of the leading unknown in the preceding equation. Such a triangular system always has a unique solution, which may be obtained by back-substitution.

The leading unknowns in the triangular form system are called *pivot* variables.

5 Elementary Operations and Equivalent Systems

The following operations used in the elimination algorithm are called *elementary row operations*, in short elementary operations.
(i) Interchange of two equations.
(ii) Addition of a constant multiple of one equation to another equation.
(iii) Multiplication of an equation by a nonzero constant c.
The system obtained from these elementary row operations is called *row-equivalent* linear system. (The term "row reduce" or simply "reduce" will mean to transform a matrix by the elementary row operations.) Since the operation of addition of equations can be undone by a corresponding subtraction and the operation of multiplication can be undone by division by c (since $c \neq 0$), the row-equivalent linear systems have the same set of solutions as the original system. The systems having the same solution sets are called *equivalent systems*.

6 Systems in Triangular and Echelon Form

`Triangular Form:`

A system of linear equation, which is reduced to the triangular form such as in Example 2, will always have an exact solution. This solution is obtained by back-substitution.

`Echelon Form, Pivot and Free Variables:`

The following system of linear equations is said to be in *row echelon form*, in short, echelon form (staircase pattern):

$$\begin{aligned} 2x_1 + 6x_2 - x_3 + 4x_4 - 2x_5 &= 15 \\ x_3 + 2x_4 + 2x_5 &= 5 \\ 3x_4 - 9x_5 &= 6 \end{aligned}$$

The leading unknown in each equation other than the first is to the right of the leading unknown in the preceding equation. The leading unknowns in the system, x_1, x_3, x_4 are called *pivot* variables and the other unknowns, x_2 and x_5 are called *free* variables.

`Echelon Matrices`

A matrix A is called a *echelon matrix*, or is said to be in *echelon form* (staircase pattern), if the following two conditions are satisfied. (The word "echelon" comes from the French term for "step of a ladder," which reflects the shape of the matrix. A *leading nonzero element* or *pivot* of a row of A is the first nonzero element in the row.)
1. All zero rows, if any, are at the bottom of the matrix.
2. Each leading nonzero entry in a row is to the right of the leading nonzero entry in the preceding row. A consequence of this rule is that all entries below pivots are zero.

An echelon matrix simplifies many tasks in linear algebra, such as solving systems of linear equations and finding a matrix's rank.

The following is an example of an echelon matrix whose pivots have been circled.

$$\begin{bmatrix} 0 & \textcircled{2} & 3 & 4 & 5 & 9 & 0 & 7 \\ 0 & 0 & 0 & \textcircled{3} & 4 & 1 & 2 & 5 \\ 0 & 0 & 0 & 0 & 0 & \textcircled{5} & 7 & 8 \\ 0 & 0 & 0 & 0 & 0 & 0 & \textcircled{8} & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

7 Determining Solutions: Gaussian Elimination

The main method for solving the general system (2) of linear equations is called *Gaussian elimination*. It essentially consists of two parts:

Part A. (Forward Elimination) Step-by-step reduction of the system yielding either a degenerate equation with no solution (which indicates the system has no solution) or an equivalent simpler system in triangular or echelon form.

Part B. (Backward Substitution) Step-by-step back-substitution to find the solution of the simpler system.

Gaussian elimination was used to solve Examples (1) and (2).

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